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# Analysis of the thermal stress behaviour of functionally graded hollow circular cylinders

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**Abstract**

This paper presents an analysis of the thermomechanical behavior of hollow circular cylinders of functionally graded material (FGM). The solutions are obtained by a novel limiting process that employs the solutions of homogeneous hollow circular cylinders, with no recourse to the basic theory or the equations of non-homogeneous thermoelasticity. Several numerical cases are studied, and conclusions are drawn regarding the general properties of thermal stresses in the FGM cylinder. We conclude that thermal stresses necessarily occur in the FGM cylinder, except in the trivial case of zero temperature. While heat resistance may be improved by sagaciously designing the material composition, careful attention must be paid to the fact that thermal stresses in the FGM cylinder are governed by more factors than are its homogeneous counterparts. The results that are presented here will serve as benchmarks for future related work.


*Keywords:* Elasticity solution; Functionally graded material; Thermal stress; Hollow circular cylinder

## 1. Introduction

Intense heat can be generated in structures during normal operation, under special conditions in emergencies, or while they are being burnt down in disasters. As an external effect on structures, heat can be as significant as directly applied forces, and can cause damage through excessive thermal stresses. Investigations into thermal stresses in elastic bodies are numerous, and the majority have been recorded in textbooks and monographs (Boley and Weiner, 1960; Barber, 1992). One of the recent focuses in the investigation of thermal stresses has been the development of new materials that can adapt to high temperature environments and tenaciously endure serious thermal stresses (Praveen and Reddy, 1998; Loy et al., 1999; Ng et al., 2000; Reddy, 2000; He et al., 2001).

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Functionally graded materials (FGMs) can be used to alleviate the harmful effects of heat on structures. FGMs are fabricated by continuously changing the volume fraction of two basic materials, usually ceramic and metal, in one direction. The FGM materials that are thus formed exhibit isotropic yet non-homogeneous thermal and mechanical properties. In the theory of elasticity, FGM materials are mostly treated as non-homogeneous materials with material constants that vary continuously along one spatial direction. Noda (1991) presented an extensive review of thermoelastic and thermo-inelastic problems.

A number of studies have dealt with thermal stresses in the basic structural components of FGMs. Shen (2001a,b, 2002a,b) has studied the thermal postbuckling of functionally graded plates and shells. Zimmerman and Lutz (1999) presented solutions to the problem of the uniform heating of a circular cylinder by the Frobenius series method. Using a perturbation approach, Obata and Noda (1994) investigated the thermal stresses in an FGM hollow sphere and in a hollow circular cylinder. Ootao and Tanigawa (1999) conducted an approximate analysis of three-dimensional thermal stresses in an FGM rectangular plate. They also discussed the optimization of the material composition of FGM hollow circular cylinders under thermal loading, based on approximate solutions of temperatures and thermal stresses (Ootao et al., 1999). Liew et al. (2001) presented an investigation of the active control of FGM plates that were subjected to a temperature gradient by the finite element method that was based on the first-order shear deformation theory. With the use of the finite element method, Reddy and Chin (1988) considered thermomechanic analysis, including the coupling effect, for FGM plates and cylinders. Tanaka et al. (1993) designed FGM property profiles using a sensitivity and optimization method that was based on the reduction of thermal stresses.

Cylindrical shells are often used as basic structural components in engineering applications. Much research has been conducted on isotropic or laminated composite plates and shells (Liew and Lim, 1995; Karunasena et al., 1995; Liew and Teo, 1998). To our knowledge, only a limited amount of work has been carried out on FGM shells. Hence, this article will develop an analytical model to deal with FGM hollow circular cylinders that are subject to the action of an arbitrary steady state or transient temperature field. Solutions are derived for the non-homogeneous thermoelasticity of steady-state temperature distributions, thermal stresses, and thermal displacements in an FGM cylinder. Numerical results and some important conclusions regarding the general properties for thermal stresses in FGM cylinders are presented and examined. These results can serve as benchmarks for future related research.

To develop the solution, the FGM cylinder is first sectioned into a number of sub-cylinders, and each of the sub-sections is approximated as homogeneous. Displacements and stresses within the homogeneous sub-cylinders are obtainable from the homogeneous solutions, from which the continuity conditions of the displacements and stresses at the interfaces can be formed. When the number of the sub-cylinders becomes infinitely large, they constitute a FGM cylinder, and the continuity conditions at the interfaces become a system of ordinary differential equations, which are solved analytically or numerically. Solutions for the FGM cylinder are therefore obtained through matching an infinitely large number of homogeneous solutions, with no recourse to the basic equations of non-homogeneous thermoelasticity.

## 2. Mathematical formulation

Steady-state temperature solutions are considered and determined for a hollow circular cylinder of FGM with an inner radius $r_0$, outer radius $r_N$, and thermal conductivity $\lambda(r) = \lambda_0 e^{pr}$, $\lambda_0$ and $p$ being constants. A cylindrical coordinate system $(r, \theta, z)$ is established for reference, with the $z$-axis lying on the axis of the cylinder. As we seek a thermal stress solution of the FGM cylinder in the plane strain condition, the temperature is independent of $z$.

To begin, we consider the temperature of axial symmetry. For the special case of $p = 0$, i.e. when the material is homogeneous, the temperature solution is

$$T(r) = F + G \log r, \tag{1}$$

where $F$ and $G$ are constants, depending on the boundary conditions of the problem.

To obtain the solution for the FGM cylinder, we proceed as follows:

(i) Section the FGM cylinder by cylindrical surfaces $r = r_1, r_2, \dots, r_{n-1}$, where $r_1 = r_0 + h$, $r_2 = r_1 + h = r_0 + 2h$, etc., to divide the whole cylinder into $n$ sub-cylinders of uniform thickness $h = (r_N - r_0)/n$.

(ii) The innermost sub-cylinder is numbered as sub-cylinder 1, the sub-cylinder next to it as sub-cylinder 2, and so on.

(iii) Assume that the non-homogeneous sub-cylinders are homogeneous, with a constant conductivity $\lambda_0 \exp(pr_{j-1})$, $r_{j-1} = r_0 + (j-1)h$, for the $j$th sub-cylinder.

(iv) The solution in Eq. (1) now applies to each of the sub-cylinders, which have been assumed to be approximately homogeneous. For the $j$th sub-cylinder, the temperature is denoted as

$$T^{(j)}(r) = F^{(j)} + G^{(j)} \log r_{j-1}, \tag{2}$$

where the superscript indicates that the affixed quantity belongs to the $j$th sub-cylinder.

(v) For a large $n$, the thickness $h$ is small, and the difference between $F^{(1)}$ and $F^{(2)}$ should be insignificant. The combination of solutions (2), $j = 1, 2, \dots, N$, should be a good approximation of the solution of the FGM cylinder, provided that the constants $F^{(1)}, G^{(1)}, F^{(2)}, G^{(2)}, \dots$ are determined first by the continuity conditions of temperatures and heat flux at the interfaces of the sub-cylinders, and finally by the boundary conditions of the FGM cylinder.

(vi) When $h \to 0$, the difference between $F^{(1)}$ and $F^{(2)}$ becomes infinitesimally small, and the two can be written in terms of a single function $F(r)$:

$$F^{(1)} = F(r_0), \quad F^{(2)} = F(r_1) \approx F(r_0) + h\left[\frac{\mathrm{d}F(r)}{\mathrm{d}r}\right]_{r=r_0}. \tag{3}$$

Similar formulas hold for $G^{(1)}$ and $G^{(2)}$ when another function, $G(r)$, is introduced, and for all other $F^{(i)}$s and $G^{(i)}$s.

(vii) All continuity conditions at the interfaces, in the form of $2(n-1)$ simultaneous algebraic equations, reduce to two simultaneous differential equations, with $F(r)$ and $G(r)$ as unknowns.

(viii) By solving the two simultaneous differential equations for $F(r)$ and $G(r)$, the exact solution for the FGM cylinder is obtained as

$$T(r) = F(r) + G(r) \log r. \tag{4}$$

In the following, all of the temperature and thermal stress solutions for the FGM cylinder are determined in accordance with the above solution scheme.

The continuity conditions of the temperatures and heat flux at $r = r_1$, which is the interface of the two innermost sub-cylinders, take the following form:

$$\begin{aligned} &F^{(2)} + G^{(2)} \log r_1 = F^{(1)} + G^{(1)} \log r_1, \\ &\lambda^{(2)} \frac{G^{(2)}}{r_1} = \lambda^{(1)} \frac{G^{(1)}}{r_1}. \end{aligned} \tag{5}$$

Eq. (5) can be rewritten as

$$\begin{aligned} &F^{(2)} = F^{(1)} + \left(1 - \frac{\lambda^{(1)}}{\lambda^{(2)}}\right) G^{(1)} \log r_1, \\ &G^{(2)} = \frac{\lambda^{(1)}}{\lambda^{(2)}} G^{(1)}. \end{aligned} \tag{6}$$

For small $h$, Eq. (6), and similar equations for $F^{(3)}$, $G^{(3)}$, etc., provide an approximate solution for the FGM cylinder. To develop an exact solution we consider the case of $h \ll 1$, for which the following formulas hold:

$$\begin{aligned}
&F^{(1)} = F(r_0), \quad G^{(1)} = G(r_0),\\
&F^{(2)} = F(r_1) \approx F(r_0) + \left[\frac{\mathrm{d}F(r)}{\mathrm{d}r}\right]_{r=r_0} h, \quad G^{(2)} = G(r_1) \approx G(r_0) + \left[\frac{\mathrm{d}G(r)}{\mathrm{d}r}\right]_{r=r_0} h,\\
&\frac{\lambda^{(1)}}{\lambda^{(2)}} = \mathrm{e}^{-ph} \approx 1 - ph,\\
&r_1 = r_0 + h = r_0\left(1 + \frac{h}{r_0}\right), \quad \frac{1}{r_1} \approx \frac{1}{r_0}\left(1 - \frac{h}{r_0}\right),
\end{aligned} \tag{7}$$

where $F(r)$ and $G(r)$ are two sufficiently smooth functions. The substitution of Eq. (7) into Eq. (6) and the application of $h \to 0$ to the resultant equations leads to the following two simultaneous differential equations:

$$\frac{\mathrm{d}F(r)}{\mathrm{d}r} = pG(r)\log r, \quad \frac{\mathrm{d}G(r)}{\mathrm{d}r} = -pG(r). \tag{8}$$

The same differential equations are obtained when we consider the limiting case of $h \to 0$ for the continuity conditions at other interfaces. The solution for $G(r)$ in Eq. (8) is easily obtainable, and that for $F(r)$ can be obtained by integration by parts. The result is as follows:

$$F(r) = \begin{cases} G_0 - F_0 E_1(pr) - F_0 \mathrm{e}^{-pr}\log r, & p > 0,\\ G_0, & p = 0,\\ G_0 + F_0 E_i(-pr) - F_0 \mathrm{e}^{-pr}\log r, & p < 0, \end{cases}$$

and

$$G(r) = \begin{cases} F_0 \mathrm{e}^{-pr}, & p > 0,\\ F_0, & p = 0,\\ F_0 \mathrm{e}^{-pr}, & p < 0, \end{cases} \tag{9}$$

where

$$E_1(pr) = \int_{pr}^{\infty} \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t, \quad E_i(-pr) = \int_{pr}^{\infty} \frac{\mathrm{e}^{-t}}{t}\,\mathrm{d}t \tag{10}$$

are exponential integrals, and have the following series expansions (Abramovitz and Stegun, 1964):

$$\begin{aligned}
&E_1(pr) = -\gamma - \log(pr) - \sum_{1}^{\infty} \frac{(-1)^n (pr)^n}{nn!},\\
&E_i(-pr) = \gamma + \log(-pr) + \sum_{1}^{\infty} \frac{(-1)^n (-pr)^n}{nn!},
\end{aligned} \tag{11}$$

with $\gamma = 0.5772156649\ldots$ being Euler's constant. $F_0$ and $G_0$ are arbitrary constants that depend on the boundary conditions of the problem.

The substitution of Eq. (9) into Eq. (4) yields the final temperature solution as follows:

$$T(r) = \begin{cases} G_0 - F_0 E_1(pr), & p > 0,\\ G_0 + F_0 \log r, & p = 0,\\ G_0 + F_0 E_i(-pr), & p < 0. \end{cases} \tag{12}$$

The fact that Eq. (12) is the exact temperature solution for the FGM cylinder can be verified by directly substituting it into the equation of heat conduction for non-homogeneous media (Tanaka et al., 1993). The two arbitrary constants that are contained in Eq. (12), i.e. $F_0$ and $G_0$, can be adjusted to satisfy an arbitrary axisymmetric distribution of temperature or heat flux in the boundary conditions for the FGM cylinder.

Turning to the solution for temperatures of $\theta$-dependence, for the special case of $p = 0$, i.e. for a homogeneous cylinder, the following solution can be obtained:

$$T(r,\theta) = \left(F_n r^n + \frac{G_n}{r^n}\right)\cos n\theta, \quad n = 1,2,3,\ldots, \tag{13}$$

where $F_n$ and $G_n$ are constants.

The solution scheme that is used above for obtaining the axisymmetric temperature distribution is again used for obtaining the $\theta$-dependent temperature distribution. The continuity conditions of the temperatures and heat flux at $r = r_1$, which is the interface of the two innermost sub-cylinders, can be written as

$$\begin{aligned} F_n^{(2)} &= \frac{1}{2}\left[\left(1 + \frac{\lambda^{(1)}}{\lambda^{(2)}}\right)F_n^{(1)} + \left(1 - \frac{\lambda^{(1)}}{\lambda^{(2)}}\right)\frac{G_n^{(1)}}{r_1^{2n}}\right], \\ G_n^{(2)} &= \frac{1}{2}\left[\left(1 - \frac{\lambda^{(1)}}{\lambda^{(2)}}\right)F_n^{(1)} r_1^{2n} + \left(1 + \frac{\lambda^{(1)}}{\lambda^{(2)}}\right)G_n^{(2)}\right]. \end{aligned} \tag{14}$$

After applying the limiting process to Eq. (14), it reduces to

$$\frac{\mathrm{d}F_n(r)}{\mathrm{d}r} = -\frac{p}{2}\left[F_n(r) - \frac{G_n(r)}{r^{2n}}\right], \quad \frac{\mathrm{d}G_n(r)}{\mathrm{d}r} = \frac{p}{2}[F_n(r)r^{2n} - G_n(r)]. \tag{15}$$

Eq. (15) is solved by the semi-inverse method (Barber, 1992). In doing so, a basic form of the solution must first be devised and proposed. Let us suppose that

$$\begin{aligned} F_n(r) &= \frac{f_{-(2n-1)}}{r^{2n-1}} + \frac{f_{-(2n-2)}}{r^{2n-2}} + \cdots + \frac{f_{-2}}{r^2} + \frac{f_{-1}}{r} + f_0 + f_1 r + f_2 r^2 + \cdots, \\ G_n(r) &= g_0 + g_1 r + g_2 r^2 + \cdots \end{aligned} \tag{16}$$

The unknown constants $f_{-(2n-1)}, f_{-(2n-2)}, \ldots, f_{-1}, f_0, \ldots$ and $g_0, g_1, \ldots$ in Eq. (16) should be fixed by first substituting Eq. (16) into Eq. (15), and then comparing and equating the coefficients of all like terms on both sides of the resultant equations. This yields

$$\begin{aligned} f_{-(2n-1)} &= -\frac{p g_0}{2(2n-1)}, \quad g_1 = -\frac{p g_0}{2}, \\ f_{-(2n-j)} &= \frac{p[g_{j-1} - f_{-(2n-j+1)}]}{2(j-2n)}, \quad j = 2,3,\ldots,2n-1, \\ f_j &= \frac{p\left(g_{2n+j-1} - f_{j-1}\right)}{2j}, \quad j = 1,2,3,\ldots, \\ g_j &= \frac{p\left(f_{-(2n-j+1)} - g_{j-1}\right)}{2j}, \quad j = 2,3,\ldots, \end{aligned} \tag{17}$$

where $f_0$ and $g_0$ are two arbitrary constants that depend on the boundary conditions of the problem. The final solution for the FGM cylinder is

$$T(r,\theta) = \left[F_n(r) r^n + \frac{G_n(r)}{r^n}\right]\cos n\theta. \tag{18}$$

The fact that solution (18) is exact can be confirmed by substituting it into the equation of heat conduction in non-homogeneous media (Tanaka et al., 1993).

Another solution is obtained when $\cos n\theta$ in Eqs. (13) and (18) is substituted with $\sin n\theta$. When $p = 0$, all of the coefficients in Eq. (16) become zero, except $f_0$ and $g_0$, which reduces solution (18) to homogeneous solution (13).

Solution (18) with factor $\cos n\theta$ and its associate with factor $\sin n\theta$, $n = 1, 2, \ldots$, make up a complete system of solutions for the FGM cylinder, and can be used to solve any particular type of boundary value problem of $\theta$-dependence.

## 3. Thermal stresses: axisymmetric temperature

The thermal stresses in the FGM hollow circular cylinder that are caused by an axisymmetric temperature distribution are to be determined. Poisson's ratio of the cylinder is constant, the coefficient of linear thermal expansion is assumed to be $\alpha = \alpha_0 \exp(qr)$, and the shear modulus is taken as $\mu = \mu_0 \exp(sr)$, where $\alpha_0$, $q$, $\mu_0$, and $s$ are material constants. It is assumed that the cylinder is in a plane strain condition and its two circular cylindrical surfaces are traction free.

As before, the FGM cylinder is initially approximated as a piecewise homogeneous cylinder. For the $j$th sub-cylinder, the temperature is given by Eq. (2):

$$T^{(j)}(r) = F^{(j)} + G^{(j)} \log r_{j-1} = \text{constant}. \tag{19}$$

Due to the uniform temperature (19), the radial stress and displacement that are induced in the $j$th sub-cylinder by the temperature alone, which is denoted with a superscript asterisk, take the form of

$$\sigma_r^{*(j)}(r) = 0, \quad u_r^{*(j)}(r) = (1 + v)\alpha^{(j)} T^{(j)} r. \tag{20}$$

Similarly, we have

$$\sigma_r^{*(j+1)}(r) = 0, \quad u_r^{*(j+1)}(r) = (1 + v)\alpha^{(j+1)} T^{(j+1)} r. \tag{21}$$

Eqs. (20) and (21) show that the stresses are continuous at the interface but the displacements are not. Consequently, additional traction systems must be set up in the sub-cylinders to eliminate the discontinuity in the radial displacements. Actually, the traction systems are thermal stresses that occur in the piecewise homogeneous cylinder. In the axisymmetric state, they can be derived from the stress function $\phi(r)$ as

$$\phi(r) = Ar^2 + B \log r. \tag{22}$$

With the use of the following general formulas for plane strain:

$$\begin{aligned}
&\sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}, \quad \sigma_\theta = \frac{\partial^2 \phi}{\partial r^2}, \quad \sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right), \quad \sigma_z = v(\sigma_r + \sigma_\theta) - \alpha E T, \\
&e_r = \frac{\partial u_r}{\partial r} = \frac{\sigma_r}{E} - \frac{v(\sigma_\theta + \sigma_z)}{E} + \alpha T, \quad e_\theta = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} = \frac{\sigma_r}{E} - \frac{v(\sigma_\theta + \sigma_z)}{E} + \alpha T, \\
&e_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) = \frac{\sigma_{r\theta}}{2\mu},
\end{aligned} \tag{23}$$

(all components of stress and strain that do not appear in Eq. (23) vanish in the present problem) and by combining the effects of the traction systems and the temperature, the conditions for the continuity of displacements and stresses at the interface can be written as

$$u_r^{(j)}(r_j) = u_r^{(j+1)}(r_j), \quad \sigma_r^{(j)}(r_j) = \sigma_r^{(j+1)}(r_j), \tag{24}$$

where

$$\begin{aligned} &u_r^{(j)}(r_j) = A^{(j)}\frac{\kappa-1}{2\mu^{(j)}}r_j - B^{(j)}\frac{1}{2\mu^{(j)}r_j} + u_r^{*(j)}(r_j), \quad \sigma_r^{(j)}(r_j) = 2A^{(j)} + \frac{B^{(j)}}{r_j^2}, \\ &u_r^{(j+1)}(r_j) = A^{(j+1)}\frac{\kappa-1}{2\mu^{(j+1)}}r_j - B^{(j+1)}\frac{1}{2\mu^{(j+1)}r_j} + u_r^{*(j+1)}(r_j), \\ &\sigma_r^{(j+1)}(r_j) = 2A^{(j+1)} + \frac{B^{(j+1)}}{r_{j+1}^2}, \end{aligned} \tag{25}$$

$\kappa = 3$–$4v$ for plane strain.

By employing Eqs. (8) and (9) via mathematical manipulations, the difference of the displacements $u_r^{*(1)}$ and $u_r^{*(2)}$ for $p > 0$ can be expressed as

$$u_r^{*(1)} - u_r^{*(2)} = -(1+v)\{\alpha(r_1)G(r_1) - F_0 q\alpha(r_1)r_1E_1(pr_1) + qG_0r_1\alpha(r_1)\}h + \cdots, \tag{26}$$

where $F_0$, $G_0$ and $G(r)$ are given in Eqs. (8) and (9). The case of $p < 0$ will be considered later.

Using Eqs. (25) and (26) in the limiting process as $h \to 0$, Eq. (24) is transformed into the following set of differential equations:

$$\begin{aligned} \frac{dA(r)}{dr} &= d_{11}A(r) + \frac{d_{12}}{r^2}B(r) + f(r), \\ \frac{dB(r)}{dr} &= d_{21}r^2A(r) + d_{22}B(r) - 2r^2f(r), \end{aligned} \tag{27}$$

where

$$\begin{aligned} &d_{11} = \frac{(\kappa-1)s}{1+\kappa}, \quad d_{12} = -\frac{s}{1+\kappa}, \quad d_{21} = -2d_{11,}, \quad d_{22} = -2d_{12,} \\ &f(r) = -2\frac{1+v}{1+\kappa}\alpha_0\mu_0 e^{wr}\left\{\frac{F_0e^{-pr}}{r} - F_0qE_1(pr) + qG_0\right\}, \quad w = q+s. \end{aligned} \tag{28}$$

To facilitate the solution of Eq. (27), we write

$$A(r) = A^*(r)\exp(wr), \quad B(r) = B^*(r)\exp(wr) \tag{29}$$

and substitute Eq. (29) into (27) to obtain

$$\widetilde{\mathbf{P}} = \widetilde{\mathbf{D}}\tilde{\mathbf{v}} + \widetilde{\mathbf{H}}, \tag{30}$$

where

$$\widetilde{\mathbf{P}} = \left[\frac{dA^*(r)}{dr}\,\frac{dB^*(r)}{dr}\right]^{\mathrm{T}}$$

$$\widetilde{\mathbf{D}} = \begin{bmatrix} d_{11}^* & \dfrac{d_{12}^*}{r^2} \\ r^2d_{12}^* & d_{22}^* \end{bmatrix}; \quad \tilde{\mathbf{v}} = [A^*(r)B^*(r)]^{\mathrm{T}}; \quad \widetilde{\mathbf{H}} = [H(r) - 2r^2H(r)]^{\mathrm{T}}$$

in which

$$
\begin{aligned}
&d_{11}^* = d_{11} - w, \quad d_{22}^* = d_{22} - w, \quad d_{12}^* = d_{12}, \quad d_{21}^* = d_{21}, \\
&H(r) = K_0\left\{c_L \log(r) + \frac{c_{-1}}{r} + c_0 + c_1 r + c_2 r^2 + \cdots\right\}, \quad K_0 = -2\frac{1+v}{1+\kappa}\alpha_0\mu_0, \\
&c_L = F_0 q, \quad c_{-1} = F_0, \quad c_0 = -pF_0 + q\gamma F_0 + qG_0 + F_0 q \log(p), \\
&c_j = \frac{F_0 q(-1)^j p^j}{j j!} + \frac{F_0(-p)^{j+1}}{(j+1)!} \quad \text{for } j = 1, 2, \ldots
\end{aligned}
\tag{31}
$$

The complete solution of Eq. (30) consists of two linearly independent homogeneous solutions and a particular solution (Tenenbaum and Pollard, 1963). We seek the particular solution by splitting it into two parts:

$$
A_p^*(r) = A_{p1}^*(r) + A_{p2}^*(r), \quad B_p^*(r) = B_{p1}^*(r) + B_{p2}^*(r). \tag{32}
$$

The first part, $A_{p1}^*(r)$ and $B_{p1}^*(r)$, is used to account for the logarithmic term in Eq. (30) that is contained in $H(r)$, while the second part, $A_{p2}^*(r)$ and $B_{p2}^*(r)$, accounts for the remaining terms of $H(r)$. For the first part we propose

$$
\begin{aligned}
&A_{p1}^*(r) = u_1 r + u_2 r^2 + \cdots + (a_1 r + a_2 r^2 + \cdots)\log(r), \\
&B_{p1}^*(r) = r^2\{v_1 r + v_2 r^2 + \cdots + (b_1 r + b_2 r^2 + \cdots)\log(r)\}.
\end{aligned}
\tag{33}
$$

Eq. (33) exactly satisfies Eq. (30), provided that we take

$$
\begin{aligned}
&a_1 = K_0 c_L, \quad u_1 = -K_0 c_L, \quad b_1 = -\frac{2}{3}K_0 c_L, \quad v_1 = \frac{2}{9}K_0 c_L, \\
&a_j = \frac{1}{j}(d_{11}^* a_{j-1} + d_{12}^* b_{j-1}), \quad u_j = \frac{1}{j}(d_{11}^* u_{j-1} + d_{12}^* v_{j-1} - a_j), \\
&b_j = \frac{1}{j+2}(d_{21}^* a_{j-1} + d_{22}^* b_{j-1}), \quad v_j = \frac{1}{j+2}(d_{21}^* u_{j-1} + d_{22}^* v_{j-1} - b_j),
\end{aligned}
\tag{34}
$$

where $j = 2, 3, \ldots$

For the second part of the particular solution it is supposed that

$$
A_{p2}^*(r) = \frac{a_{-1}}{r} + a_0 + a_1 r + a_2 r^2 + \cdots, \quad B_{p2}^*(r) = b_0 + b_1 r + b_2 r^2 + \cdots \tag{35}
$$

It can be shown that Eq. (30) is exactly satisfied by Eq. (35), providing that

$$
\begin{aligned}
&a_{-1} = \frac{K_0 c_{-1}}{d_{22}^* - d_{11}^*}, \quad a_0 = 0, \quad b_0 = -\frac{a_{-1}}{d_{12}^*}, \quad b_1 = d_{22}^* b_0, \\
&b_j = \frac{1}{j}(d_{21}^* a_{j-3} + d_{22}^* b_{j-1} - 2K_0 c_{j-3}) \quad \text{for } j = 2, 3, \ldots, \\
&a_j = \frac{1}{j}(d_{11}^* a_{j-1} + d_{12}^* b_{j+1} + K_0 c_{j-1}) \quad \text{for } j = 1, 2, \ldots
\end{aligned}
\tag{36}
$$

This concludes the development of the particular solution. The particular solution exactly determines the effect of the temperature on the deformation of the FGM cylinder, which causes tractions on the cylindrical surfaces that in general do not vanish. The two homogeneous solutions to Eq. (30) with $H(r) = -2r^2 H(r) = 0$ should be added to the particular solution to cancel the redundant tractions. The thermo-elasticity solution of the axisymmetric problem for the FGM cylinder is obtained as

$$u_r = \frac{(\kappa - 1)A(r)r}{2\mu(r)} - \frac{B(r)}{2\mu(r)r} + \alpha(r)(1+v)[F(r) + G(r)\log r]r, \quad u_\theta = u_z = 0,$$
$$\sigma_r = 2A(r) + \frac{B(r)}{r^2}, \quad \sigma_\theta = 2A(r) - \frac{B(r)}{r^2}, \tag{37}$$
$$\sigma_z = v(\sigma_r + \sigma_\theta) - 2\alpha(r)\mu(r)(1+v)[F(r) + G(r)\log r], \quad \sigma_{zr} = \sigma_{r\theta} = \sigma_{\theta z} = 0,$$

where

$$A(r) = A_1(r) + A_2(r) + A_p(r), \quad B(r) = B_1(r) + B_2(r) + B_p(r), \tag{38}$$

where $A_1(r)$, $A_2(r)$, $B_1(r)$ and $B_2(r)$ are the homogeneous solutions.

The fact that Eq. (37) is exact follows from the solution procedure and the derivation. The exactness can also be proven as follows. The three displacement components in Eq. (37) are continuous and single-valued. From them the six stress components can be derived by Eq. (23), Hooke's law, and the result is as shown in Eq. (37). That these stress components satisfy the equations of equilibrium can be confirmed by directly substituting them into the equations.

When $p < 0$, $E_1(pr)$ in Eqs. (26) and (28) should be replaced with $E_1(-pr)$, whereas $c_0$ and $c_j$ in Eq. (31) should be rewritten as

$$c_0 = -pF_0 + q\gamma F_0 + qG_0 + F_0 q\log(-p),$$
$$c_j = \frac{F_0 q(-1)^j(-p)^j}{jj!} + \frac{F_0(-p)^{j+1}}{(j+1)!}. \tag{39}$$

Elsewhere, the result remains unchanged.

## 4. Thermal stresses: temperatures of $\theta$-dependence

The thermal stresses in the FGM cylinder that are induced by the temperature field (18) for $n = 1, 2, 3, \ldots$ are sought. First we treat the case of $n = 1$ separately. After that, the cases of $n = 2, 3, \ldots$ will be treated in a unified manner.

### 4.1. Solution for n = 1

As an initial step to deal with the case of $n = 1$, consider a homogeneous cylinder that is affected by the temperature field (Eq. (13))

$$T(r,\theta) = \left[F_1 r + \frac{G_1}{r}\right]\cos\theta, \tag{40}$$

where $F_1$ and $G_1$ are constants. Except for $\sigma_z$, the temperature (40) does not bring about thermal stresses in homogeneous cylinders (Boley and Weiner, 1960). Each sub-cylinder in the piecewise homogeneous cylinder is free from $\sigma_r$, $\sigma_\theta$, and $\sigma_{r\theta}$. At the interfaces, however, displacement discontinuity generally occurs due to different thermal expansions in different sub-cylinders. Traction systems at the interfaces are thus set up to eliminate the discontinuity in displacements in both the $r$ and the $\theta$ directions. For $n = 1$, these traction systems can be deduced from the following stress function:

$$\phi(r,\theta) = Ar^3\cos\theta + Br\log r\cos\theta + Cr\theta\sin\theta + \frac{D}{r}\cos\theta. \tag{41}$$

It is known that displacements $u_r$ and $u_\theta$ are derived from $B\log r\cos\theta$, and $Cr\theta\sin\theta$ in Eq. (41) are multi-valued. The elimination of the multi-valuedness necessitates

$$C = \kappa_0 B, \quad \kappa_0 = \frac{1+\kappa}{1-\kappa}. \tag{42}$$

Consider the continuity conditions of the stresses and displacements on any one of the interfaces, for instance the innermost interface. Using Eqs. (23), (40), (41)–(42), the continuity conditions can be written as

$$u_r^{(2)}(r_1,\theta) = u_r^{(1)}(r_1,\theta), \tag{43}$$

where $u_r^{(a)}(r_1,\theta)$ ($a = 1$ or 2) can be expressed as

$$u_r^{(a)}(r_1,\theta) = \left\{ A^{(a)} \frac{\kappa - 2}{2\mu^{(2)}} r_1^2 + \frac{B^{(a)}}{4\mu^{(a)}} \{[(1+\kappa_0)\kappa - 1 + \kappa_0]\log r_1 - 1 - \kappa_0\} + \frac{D^{(a)}}{2\mu^{(a)} r_1^2} \right.$$
$$\left. + (1+v)\alpha^{(2)} \left( \frac{F^{(a)} r_1^2}{2} + G^{(a)} \log r_1 \right) + U^{(a)} \right\} \cos\theta, \tag{44}$$

and

$$u_\theta^{(2)}(r_1,\theta) = u_\theta^{(1)}(r_1,\theta), \tag{45}$$

where $u_\theta^{(a)}(r_1,\theta)$ ($a = 1$ or 2) can be expressed as

$$u_\theta^{(a)}(r_1,\theta) = \left\{ A^{(a)} \frac{\kappa + 2}{2\mu^{(a)}} r_1^2 + \frac{B^{(a)}}{4\mu^{(a)}} \{[1 - \kappa_0 - (1+\kappa_0)\kappa]\log r_1 - 1 - \kappa_0\} + \frac{D^{(a)}}{2\mu^{(a)} r_1^2} \right.$$
$$\left. + (1+v)\alpha^{(a)} \left[ \frac{F^{(a)} r_1^2}{2} - G^{(a)}(\log r_1 + 1) - U^{(a)} \right] \right\} \sin\theta. \tag{46}$$

In Eqs. (44) and (46), $U^{(1)}$ and $U^{(2)}$ are rigid-body displacements that have no effect on strains and stresses, and they will be ignored in the further development. Furthermore,

$$\sigma_r^{(2)}(r_1,\theta) = \sigma_r^{(1)}(r_1,\theta), \tag{47}$$

where $\sigma_r^{(a)}(r_1,\theta)$ ($a = 1$ or 2) can be expressed as

$$\sigma_r^{(a)}(r_1,\theta) = \left[ 2A^{(a)} r_1 + \frac{(1+2\kappa_0)B^{(a)}}{r_1} - \frac{2D^{(a)}}{r_1^3} \right] \cos\theta, \tag{48}$$

and

$$\sigma_{r\theta}^{(2)}(r_1,\theta) = \sigma_{r\theta}^{(1)}(r_1,\theta), \tag{49}$$

where $\sigma_{r\theta}^{(a)}(r_1,\theta)$ ($a = 1$ or 2) can be expressed as

$$\sigma_{r\theta}^{(a)}(r_1,\theta) = \left[ 2A^{(a)} r_1 + \frac{B^{(a)}}{r_1} - \frac{2D^{(a)}}{r_1^3} \right] \sin\theta. \tag{50}$$

Eqs. (47)–(50) clearly show that continuity conditions (47) and (49) can be satisfied only when $B^{(2)} = B^{(1)}$. Further analysis of stress continuity at other interfaces confirms that all $B^{(j)}$ in the piecewise homogeneous cylinder must be a constant: $B^{(1)} = B^{(2)} = \cdots = B^{(n)} = B_0 = \text{constant}$. Consequently, Eqs. (48) and (50) reduce to

$$A^{(2)} - \frac{D^{(2)}}{r_1^4} = A^{(1)} - \frac{D^{(1)}}{r_1^4}. \tag{51}$$

Eqs. (43), (45), and (51) can be used to determine $A^{(2)}$, $D^{(2)}$ and $U^{(2)}$ in terms of each others. Consider Eq. (51) and the equation that is obtained by adding (43)–(45). These two equations do not contain $U^{(2)}$ and $U^{(1)}$. By applying the limiting condition, $h \to 0$, to the two equations, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}A(r)}{\mathrm{d}r} = c_1 A(r) + c_2 \frac{D(r)}{r^4} + c_3 \frac{B_0}{r^2} + \frac{(1+v)\mu(r)\alpha(r)}{1+\kappa}\left\{\frac{1}{r^2}\left[\frac{\mathrm{d}G_1(r)}{\mathrm{d}r} + qG_1(r)\right] - \frac{\mathrm{d}F_1(r)}{\mathrm{d}r} - qF_1(r)\right\},\\
&\frac{\mathrm{d}D(r)}{\mathrm{d}r} = r^4 \frac{\mathrm{d}A(r)}{\mathrm{d}r},
\end{aligned} \tag{52}$$

where

$$c_1 = \frac{\kappa s}{1+\kappa}, \quad c_2 = \frac{s}{1+\kappa}, \quad c_3 = -\frac{s(1+\kappa_0)}{2(1+\kappa)}. \tag{53}$$

For the particular solution of Eq. (52), we propose that

$$A_p(r) = A^\circ(r)\mathrm{e}^{wr}, \quad D_p(r) = D^\circ(r) r^4 \mathrm{e}^{wr}, \quad B_{0P} = 0. \tag{54}$$

The substitution of Eq. (54) into (52) yields

$$\begin{aligned}
&\frac{\mathrm{d}A^\circ(r)}{\mathrm{d}r} = c_1^* A^\circ(r) + c_2 D^\circ(r) + \left\{\frac{h_{-2}}{r^2} + \frac{h_{-1}}{r} + h_0 + h_1 r + h_2 r^2\right\},\\
&\frac{\mathrm{d}D^\circ(r)}{\mathrm{d}r} = c_1 A^\circ(r) + \left(c_2^* - \frac{4}{r}\right) D^\circ(r) + \left\{\frac{h_{-2}}{r^2} + \frac{h_{-1}}{r} + h_0 + h_1 r + h_2 r^2\right\},
\end{aligned} \tag{55}$$

where

$$\begin{aligned}
&c_1^* = c_1 - w, \quad c_2^* = c_2 - w,\\
&h_{-2} = c_4(qg_0 + g_1 + f_{-1}), \quad h_{-1} = c_4(qg_1 + 2g_2 - qf_{-1}), \quad h_0 = c_4(qg_2 + 3g_3 - qf_0 - f_1),\\
&h_j = c_4[qg_{j+2} + (j+3)g_{j+3} - qf_j - (j+1)f_{j+1}], \quad j = 1, 2, \ldots,\\
&c_4 = \frac{(1+v)\alpha_0\mu_0}{1+\kappa}.
\end{aligned} \tag{56}$$

To find a particular solution, $A_p^\circ(r)$ and $D_p^\circ(r)$, to Eq. (55), we suppose that

$$\begin{aligned}
&A_p^\circ(r) = \frac{a_{-3}}{r^3} + \frac{a_{-2}}{r^2} + \frac{a_{-1}}{r} + a_0 + a_1 r + a_2 r^2 + \cdots,\\
&D_p^\circ(r) = \frac{d_{-4}}{r^4} + \frac{d_{-3}}{r^3} + \frac{d_{-2}}{r^2} + \frac{d_{-1}}{r} + d_0 + d_1 r + d_2 r^2 + \cdots
\end{aligned} \tag{57}$$

It is confirmed that Eq. (55) is exactly satisfied by Eq. (57), provided that we take

$$\begin{aligned}
&a_{-3} = -\frac{c_2 d_{-4}}{3}, \quad d_{-3} = c_2^* d_{-4}, \quad a_{-2} = -\frac{c_1^* a_{-3} + c_2 d_{-3}}{2}, \quad d_{-2} = \frac{c_1 a_{-3} + c_2^* d_{-3}}{2},\\
&a_{-1} = -(c_1^* a_{-2} + c_2 d_{-2} + h_{-2}), \quad d_{-1} = \frac{c_1 a_{-2} + c_2^* d_{-2} + h_{-2}}{3},
\end{aligned} \tag{58}$$

In Eq. (58), $a_{-3}$, $d_{-3}$, $a_{-2}$, $d_{-2}$, $a_{-1}$, and $d_{-1}$ are determined in terms of $d_{-4}$, the value of which can be fixed by the following equation:

$$c_1^* a_{-1} + c_2 d_{-1} + h_{-1} = 0. \tag{59}$$

Eq. (59) is a linear algebraic equation for $d_{-4}$, which can be exactly solved as

$$\begin{aligned}
&a_0 = 0, \quad d_0 = \frac{1}{4}(c_1 a_{-1} + c_2^* d_{-1} - 4d_0 + h_{-1}),\\
&a_j = \frac{c_1^* a_{j-1} + c_2(d_{j-1} - 4d_0 + h_j)}{j}, \quad d_j = \frac{c_1 a_{j-1} + c_2^*(d_{j-1} - 4d_j + h_j)}{j}, \quad j = 1, 2, \ldots
\end{aligned} \tag{60}$$

With the use of Eq. (54), the particular solution to Eq. (52), $A_p(r)$ and $D_p(r)$, can be obtained directly from $A^\circ(r)$ and $D^\circ(r)$.

Finally, the thermal stresses in the FGM cylinder can be expressed, in terms of $A(r) = A_h(r) + A_p(r)$, $D(r) = D_h(r) + D_p(r)$ and $B_0 = B_{0h} + B_{0p}$, where the homogeneous solutions $A_h(r)$, $D_h(r)$, and $B_{0h}$ are given as follows:

$$\begin{aligned}
\sigma_r &= \left[2A(r)r + \frac{(1+2\kappa_0)B_0}{r} - \frac{2D(r)}{r^3}\right]\cos\theta, \\
\sigma_\theta &= \left[6A(r)r + \frac{B_0}{r} + \frac{2D(r)}{r^3}\right]\cos\theta, \quad \sigma_{r\theta} = \left[2A(r)r + \frac{B_0}{r} - \frac{2D(r)}{r^3}\right]\sin\theta, \\
\sigma_z &= v(\sigma_r + \sigma_\theta) - 2\alpha(r)\mu(r)(1+v)\left[F_1(r)r + \frac{G_1(r)}{r}\right], \quad \sigma_{zr} = \sigma_{\theta z} = 0.
\end{aligned} \tag{61}$$

That solution (61) is exact can be confirmed in a similar manner to the confirmation for solution (37).

### 4.2. Solution for $n = 2, 3, \ldots$

Now turn to the case of $n = 2, 3, \ldots$ in Eq. (18). To begin, we consider a homogeneous cylinder that is subjected to the temperature field

$$T(r,\theta) = \left[F_n r^n + \frac{G_n}{r^n}\right]\cos n\theta, \quad n = 2, 3, \ldots, \tag{62}$$

where $F_n$ and $G_n$ are constants. The temperature (62) does not bring about thermal stresses in the piecewise homogeneous cylinder, except for $\sigma_z$ (Boley and Weiner, 1960). As in the case of $n = 1$, however, traction systems on the interfaces of the piecewise homogeneous cylinder are set up to eliminate the discontinuity in displacements at the interfaces. These traction systems can be deduced from the following stress function:

$$\phi(r,\theta) = \left(Ar^{n+2} + \frac{B}{r^{n-2}} + Cr^n + \frac{D}{r^n}\right)\cos n\theta. \tag{63}$$

By using stress function (63) and (23), the continuity conditions that apply at all of the interfaces can be written down. For instance, for the innermost interface we have

$$u_r^{(2)}(r_1,\theta) = u_r^{(1)}(r_1,\theta), \tag{64}$$

where $u_r^{(a)}(r_1,\theta)$ ($a = 1$ or 2) can be expressed as

$$\begin{aligned}
u_r^{(a)}(r_1,\theta) = &\left\{A^{(a)}\frac{\kappa - n - 1}{2\mu^{(2)}}r_1^{n+1} + B^{(a)}\frac{\kappa + n - 1}{2\mu^{(a)}r_1^{n-1}} - C^{(a)}\frac{nr_1^{n-1}}{2\mu^{(a)}} + D^{(a)}\frac{n}{2\mu^{(a)}r_1^{n+1}}\right. \\
&\left. + (1+v)\alpha^{(a)}\left(\frac{F^{(a)}r_1^{n+1}}{n+1} - \frac{G^{(a)}}{(n-1)r_1^{n-1}}\right)\right\}\cos n\theta,
\end{aligned} \tag{65}$$

and

$$u_\theta^{(2)}(r_1,\theta) = u_\theta^{(1)}(r_1,\theta), \tag{66}$$

where $u_\theta^{(a)}(r_1,\theta)$ ($a = 1$ or 2) can be expressed as

$$u_\theta^{(a)}(r_1,\theta)=\left\{A^{(a)}\frac{\kappa+n+1}{2\mu^{(a)}}r_1^{n+1}+B^{(a)}\frac{n-\kappa-1}{2\mu^{(a)}r_1^{n-1}}+C^{(a)}\frac{nr_1^{n-1}}{2\mu^{(a)}}+D^{(a)}\frac{n}{2\mu^{(a)}r_1^{n+1}}\right.$$
$$\left.+(1+v)\alpha^{(a)}\left(\frac{F^{(a)}r_1^{n+1}}{n+1}+\frac{G^{(a)}}{(n-1)r_1^{n-1}}\right)\right\}\sin n\theta, \tag{67}$$

and

$$\sigma_r^{(2)}(r_1,\theta)=\sigma_r^{(1)}(r_1,\theta), \tag{68}$$

where $\sigma_r^{(a)}(r_1,\theta)$ ($a=1$ or 2) can be expressed as

$$\sigma_r^{(a)}(r_1,\theta)=\left[A^{(a)}(2-n)(n+1)r_1^n+B^{(a)}\frac{(1-n)(n+2)}{r_1^n}+C^{(a)}n(1-n)r_1^{n-2}-D^{(a)}\frac{n(n+1)}{r^{n+2}}\right]\cos n\theta, \tag{69}$$

and

$$\sigma_{r\theta}^{(2)}(r_1,\theta)=\sigma_{r\theta}^{(1)}(r_1,\theta), \tag{70}$$

where $\sigma_{r\theta}^{(a)}(r_1,\theta)$ ($a=1$ or 2) can be expressed as

$$\sigma_{r\theta}^{(a)}(r_1,\theta)=\left[n(n+1)r_1^nA^{(a)}+\frac{n(1-n)}{r_1^n}B^{(a)}+n(n-1)r_1^{n-2}C^{(a)}-\frac{n(n+1)}{r_1^{n+2}}D^{(a)}\right]\sin n\theta. \tag{71}$$

Through the limiting procedure $h\to 0$, Eqs. (64), (66), (68), and (70) become a system of differential equations as

$$\mathbf{P}=\mathbf{C}\mathbf{v}+\mathbf{H}, \tag{72}$$

where

$$\mathbf{P}=\begin{bmatrix}\frac{dA(r)}{dr} & \frac{dB(r)}{dr} & \frac{dC(r)}{dr} & \frac{dD(r)}{dr}\end{bmatrix}^{\mathrm{T}},$$

$$\mathbf{C}=\begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14}\\ c_{21} & c_{22}-\frac{2n}{r} & c_{23} & c_{24}\\ c_{31} & c_{32} & c_{33}-\frac{2}{r} & c_{34}\\ c_{41} & c_{42} & c_{43} & c_{44}-\frac{2n+2}{r}\end{bmatrix},$$

$$\mathbf{v}=[A(r)\quad B(r)\quad C(r)\quad D(r)]^{\mathrm{T}},$$

$$\mathbf{H}=[H_1(r)\quad H_2(r)\quad H_3(r)\quad H_4(r)]^{\mathrm{T}}$$

in which

$$\mathbf{C}=\begin{bmatrix} \dfrac{s\kappa}{1+\kappa} & -\dfrac{(1-n)s}{1+\kappa} & 0 & \dfrac{ns}{1+\kappa} \\ -\dfrac{(n+1)s}{1+\kappa} & \dfrac{s\kappa}{1+\kappa}-\dfrac{2n}{r} & -\dfrac{ns}{1+\kappa} & 0 \\ \dfrac{(n+1)(1-\kappa)s}{n(1+\kappa)} & -\dfrac{(n^2-1+\kappa)s}{n(1+\kappa)} & \dfrac{s}{1+\kappa}-\dfrac{2}{r} & -\dfrac{(n+1)s}{1+\kappa} \\ \dfrac{(n^2-1+\kappa)s}{n(1+\kappa)} & \dfrac{(1-n)(\kappa-1)s}{n(1+\kappa)} & -\dfrac{(1-n)s}{1+\kappa} & \dfrac{s}{1+\kappa}-\dfrac{2n+2}{r} \end{bmatrix},$$

$$\mathbf{H}=\left[-\frac{\widetilde{E}\widetilde{F}}{(n+1)} \quad \frac{\widetilde{E}\widetilde{G}}{(n-1)} \quad \frac{\widetilde{E}}{n}\left\{r^2\widetilde{F}-\frac{\widetilde{G}}{(n-1)r^{2n-2}}\right\} \quad -\frac{\widetilde{E}}{n}\left\{\frac{r^{2n+2}}{1+n}\widetilde{F}+r^2\widetilde{G}\right\}\right]^{\mathrm{T}}, \tag{73}$$

$$\widetilde{E}=\frac{2(1+v)}{(1+\kappa)}\alpha(r)\mu(r), \quad \widetilde{F}=qF(r)+\frac{\mathrm{d}F(r)}{\mathrm{d}r}, \quad \widetilde{G}=qG(r)+\frac{\mathrm{d}G(r)}{\mathrm{d}r}.$$

To facilitate the solution of the simultaneous differential equations (72), we put

$$A(r)=A^{\circ}(r)\mathrm{e}^{wr}, \quad B(r)=B^{\circ}(r)r^{2n}\mathrm{e}^{wr}, \quad C(r)=C^{\circ}(r)r^{2}\mathrm{e}^{wr}, \quad D(r)=D^{\circ}(r)r^{2n+2}\mathrm{e}^{wr}, \quad w=q+s. \tag{74}$$

To recast Eqs. (72) into the following form:

$$\mathbf{P}^{\circ}=\mathbf{C}^{\circ}\mathbf{v}^{\circ}+\mathbf{H}^{\circ}, \tag{75}$$

where

$$\mathbf{P}^{\circ}=\left[\frac{\mathrm{d}A^{\circ}(r)}{\mathrm{d}r} \quad \frac{\mathrm{d}B^{\circ}(r)}{\mathrm{d}r} \quad \frac{\mathrm{d}C^{\circ}(r)}{\mathrm{d}r} \quad \frac{\mathrm{d}D^{\circ}(r)}{\mathrm{d}r}\right]^{\mathrm{T}},$$

$$\mathbf{C}^{*}=\begin{bmatrix} c_{11}^{*} & c_{12}^{*} & c_{13}^{*} & c_{14}^{*} \\ c_{21}^{*} & c_{22}^{*}-\dfrac{2n}{r} & c_{23}^{*} & c_{24}^{*} \\ c_{31}^{*} & c_{32}^{*} & c_{33}^{*}-\dfrac{2}{r} & c_{34}^{*} \\ c_{41}^{*} & c_{42}^{*} & c_{43}^{*} & c_{44}^{*}-\dfrac{2n+2}{r} \end{bmatrix},$$

$$\mathbf{v}^{\circ}=[A^{\circ}(r) \quad B^{\circ}(r) \quad C^{\circ}(r) \quad D^{\circ}(r)]^{\mathrm{T}},$$

$$\mathbf{H}^{\circ}=[H_1^{\circ}(r) \quad H_2^{\circ}(r) \quad H_3^{\circ}(r) \quad H_4^{\circ}(r)]^{\mathrm{T}}$$

in which

$$c^{*}(i,j)=c(i,j), \ \ i\neq j, \quad c^{*}(i,j)=c(i,j)-w, \ \ i=j,$$

$$\mathbf{H}^{\circ}=\left[-\frac{\widetilde{E}^{\circ}\widetilde{F}}{(n+1)} \quad \frac{\widetilde{E}^{\circ}\widetilde{G}}{r^{2n}(n-1)} \quad \frac{\widetilde{E}^{\circ}}{nr^{2}}\left\{r^2\widetilde{F}-\frac{\widetilde{G}}{(n-1)r^{2n-2}}\right\} \quad -\frac{\widetilde{E}^{\circ}}{nr^{2n+2}}\left\{\frac{(r^{2n+2})\widetilde{F}}{1+n}+r^2\widetilde{G}\right\}\right]^{\mathrm{T}}, \tag{76}$$

$$\widetilde{E}^{\circ}(r)=\frac{2(1+v)}{(1+\kappa)}\alpha_0\mu_0.$$

Using Eqs. (16) and (17), $H_j^\circ(r)$, $j = 1, 2, 3, 4$, can be expanded into the series form

$$\mathbf{H}^\circ = [\,\Theta_1^\circ(r) \quad \Theta_2^\circ(r) \quad \Theta_3^\circ(r) \quad \Theta_4^\circ(r)\,]^{\mathrm{T}},$$

where

$$\begin{aligned}
\Theta_\alpha^\circ(r) &= \frac{h_{\alpha,-2n}}{r^{2n}} + \frac{h_{\alpha,-(2n-1)}}{r^{2n-1}} + \cdots + \frac{h_{\alpha,-2}}{r^2} + \frac{h_{\alpha,-1}}{r} + h_{\alpha,0} + h_{\alpha,1}r + h_{\alpha,2}r^2 + \cdots, \\
h_{1,j} &= -\frac{\widetilde{E}^\circ}{(n+1)}[qf_j + (j+1)f_{j+1}], \quad f_{-2n} = 0, \\
h_{2,j} &= \frac{\widetilde{E}^\circ}{(n-1)}[qg_{j+2n} + (j+2n+1)g_{j+2n+1}], \\
h_{3,j} &= -\frac{1}{n}[h_{2,j} + (n+1)h_{1,j}], \\
h_{4,j} &= \frac{1}{n}[h_{1,j} - (n-1)h_{2,j}]
\end{aligned} \tag{77}$$

in which $j = -2n, -(2n-1), \ldots, -2, -1, 0, 1, 2, \ldots$

To find the particular solution for Eqs. (75), we propose that

$$\begin{aligned}
A_p^\circ(r) &= \frac{a_{-(2n-1)}}{r^{2n-1}} + \frac{a_{-(2n-2)}}{r^{2n-2}} + \cdots + \frac{a_{-2}}{r^2} + \frac{a_{-1}}{r} + a_0 + a_1 r + \cdots, \\
B_p^\circ(r) &= \frac{b_{-2n}}{r^{2n}} + \frac{b_{-(2n-1)}}{r^{2n-1}} + \frac{b_{-(2n-2)}}{r^{2n-2}} + \cdots + \frac{b_{-2}}{r^2} + \frac{b_{-1}}{r} + b_0 + b_1 r + \cdots, \\
C_p^\circ(r) &= \frac{c_{-(2n-1)}}{r^{2n-1}} + \frac{c_{-(2n-2)}}{r^{2n-2}} + \cdots + \frac{c_{-2}}{r^2} + \frac{c_{-1}}{r} + c_0 + c_1 r + \cdots, \\
D_p^\circ(r) &= \frac{d_{-(2n-1)}}{r^{2n-1}} + \frac{d_{-(2n-2)}}{r^{2n-2}} + \cdots + \frac{d_{-2}}{r^2} + \frac{d_{-1}}{r} + d_0 + d_1 r + \cdots
\end{aligned} \tag{78}$$

By substituting Eq. (78) into (75) and the equating of the coefficients of like terms on both sides of the resulting equations, the following formulas are obtained:

$$\begin{aligned}
& a_{-(2n-1)} = -\frac{c_{12}^* b_{-2n} + h_{1,-2n}}{2n-1}, \quad b_{-(2n-1)} = c_{22}^* b_{-2n} + h_{2,-2n}, \\
& c_{-(2n-1)} = -\frac{c_{32}^* b_{-2n} + h_{3,-2n}}{2n-3}, \quad d_{-(2n-1)} = \frac{c_{42}^* b_{-2n} + h_{4,-2n}}{3}, \\
& \Xi_\alpha = c_{\alpha1}^* a_{-(2n-j+1)} + c_{\alpha2}^* b_{-(2n-j+1)} + c_{\alpha3}^* c_{-(2n-j+1)} + c_{\alpha4}^* d_{-(2n-j+1)} + h_{\alpha,-(2n-j+1)}, \\
& a_{-(2n-j)} = -\frac{\Xi_1}{2n-j}, \quad b_{-(2n-j)} = \frac{\Xi_2}{j}, \quad c_{-(2n-j)} = -\frac{\Xi_3}{2n-j-2}, \quad d_{-(2n-j)} = \frac{\Xi_4}{j+2}, \\
& \Psi_{\alpha,\beta} = c_{\alpha1}^* a_{-\beta} + c_{\alpha2}^* b_{-\beta} + c_{\alpha3}^* c_{-\beta} + c_{\alpha4}^* d_{-\beta} + h_{\alpha,-\beta}, \\
& a_{-2} = -\frac{\Psi_{1,3}}{2}, \quad b_{-2} = \frac{\Psi_{2,3}}{2n-2}, \quad d_{-2} = \frac{\Psi_{4,3}}{2n},
\end{aligned} \tag{79}$$

where $j = 2, 3, \ldots, 2n-3$.

In Eq. (79), the coefficients $a_j$, $b_j$, $c_j$, and $d_j$ are expressed exactly in terms of a constant $b_{-2n}$, the exact value of which can be determined by the following equation:

$$c_{31}^* a_{-3} + c_{32}^* b_{-3} + c_{33}^* c_{-3} + c_{34}^* d_{-3} + h_{3,-3} = 0. \tag{80}$$

Eq. (80) is a linear algebraic equation for $b_{-2n}$, which can be exactly solved as

$$a_{-1} = -\Psi_{1,2}, \quad b_{-1} = \frac{\Psi_{2,2}}{2n-1}, \quad c_{-1} = \Psi_{3,2}, \quad d_{-1} = \frac{\Psi_{4,2}}{2n+1}, \tag{81}$$

$$a_0 = 0, \quad b_0 = \frac{\Psi_{2,1}}{2n}, \quad c_0 = \frac{\Psi_{3,1}}{2}, \quad d_0 = \frac{\Psi_{4,1}}{2n+2}. \tag{82}$$

The coefficients that are expressed in Eqs. (81) and (82) are given in terms of a constant $c_{-2}$, the exact value of which can be determined by the following equation:

$$c_{11}^* a_{-1} + c_{12}^* b_{-1} + c_{13}^* c_{-1} + c_{14}^* d_{-1} + h_{1,-1} = 0. \tag{83}$$

Eq. (83) is a linear algebraic equation for $c_{-2}$, which can be solved exactly:

$$a_j = \frac{\Psi_{1,j-1}}{j}, \quad b_j = \frac{\Psi_{2,j-1}}{2n+j}, \quad c_j = \frac{\Psi_{3,j-1}}{2+j}, \quad d_j = \frac{\Psi_{4,j-1}}{2n+2+j}, \tag{84}$$

where $j = 1, 2, 3, \ldots$

The particular solution to Eq. (72), $A_p(r)$, etc., can be obtained from solution (78) using Eq. (74). The thermal stresses in the FGM cylinder can be expressed, in terms of $A(r) = A_h(r) + A_p(r)$, $B(r) = B_h(r) + B_p(r)$, $C(r) = C_h(r) + C_p(r)$, and $D(r) = D_h(r) + D_p(r)$, as follows:

$$\begin{aligned}
\sigma_r &= \left[(n+1)(2-n)A(r)r^n + \frac{(1-n)(n+2)B(r)}{r^n} + n(1-n)C(r)r^{n-2} - \frac{n(1+n)D(r)}{r^{n+2}}\right]\cos\theta, \\
\sigma_{r\theta} &= \left[n(n+1)A(r)r^n + \frac{n(1-n)B(r)}{r^n} + n(n-1)C(r)r^{n-2} - \frac{n(1+n)D(r)}{r^{n+2}}\right]\sin\theta, \\
\sigma_z &= v(\sigma_r + \sigma_\theta) - 2\alpha(r)\mu(r)(1+v)\left[F_n(r)r + \frac{G_n(r)}{r}\right], \quad \sigma_{zr} = \sigma_{\theta z} = 0.
\end{aligned} \tag{85}$$

That solution (85) is exact can be confirmed in a manner that is similar to the confirmation for solution (37).

Temperatures (40) and (62) are even functions in $\theta$, and induce displacements and stresses that are symmetric to the polar axis $\theta = 0$. Another type of temperature distribution, when $\cos\theta$ in (40) and (62) is replaced with $\sin\theta$, induces displacements and stresses that are antisymmetric with respect to the polar axis. Exact expressions for these displacements and stresses can be obtained by a procedure that is quite similar to that which is used for cos $\theta$, and the details are therefore omitted.

## 5. Results and discussion

### 5.1. Temperature distribution brings about no thermal stresses

In the axisymmetric case, Eq. (12) shows that $T(r) = T_0 =$ constant is a temperature solution for the FGM cylinder. Unlike homogeneous materials that can expand freely in a constant temperature field, the expansion of FGM materials is constrained, and large values of $\sigma_r$ and $\sigma_\theta$ can be generated, as shown in Fig. 1. The thermal stresses $\sigma_r$ and $\sigma_\theta$ vanish only when $q = 0$. However, for $q = 0$, $\alpha(r) = \alpha_0 =$ constant and the FGM cylinder is homogeneous with respect to thermal expansion.

When the temperature is $\theta$-dependent, thermal stresses $\sigma_r$, $\sigma_{r\theta}$, and $\sigma_\theta$ occur in the FGM cylinder even when $q = 0$, as shown in Fig. 2. This is because temperature solution (18) is no longer a harmonic function of the homogeneous temperature solution, due to the $r$-dependence of the thermal conductivity $\lambda(r)$. Only when $q$ and $p$ are both zero can the FGM cylinder be without thermal stresses $\sigma_r$, $\sigma_{r\theta}$, and $\sigma_\theta$, i.e. only when the FGM cylinder is de facto homogeneous with respect to both thermal expansion and thermal conductance.

An important problem in thermoelasticity is that the temperature field that does not induce thermal stresses (Boley and Weiner, 1960). It follows from the above analysis that for a FGM cylinder the only

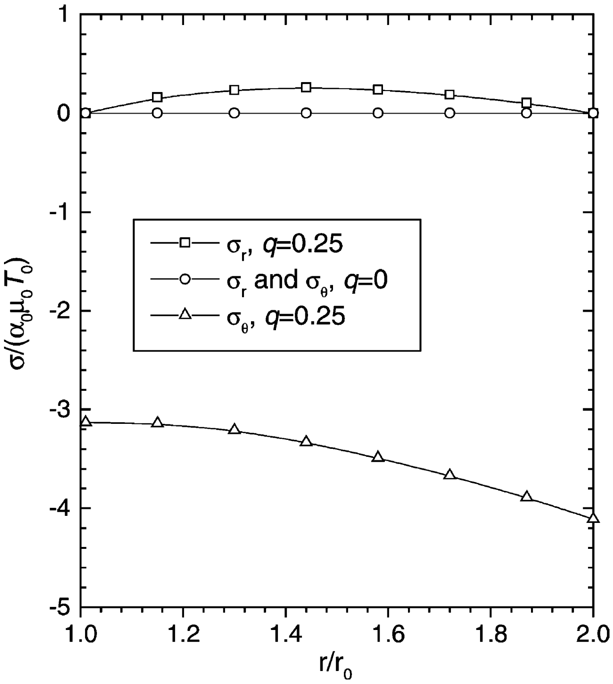


Fig. 1. Thermal stresses induced by uniform temperature ($q = 0, 0.25$).

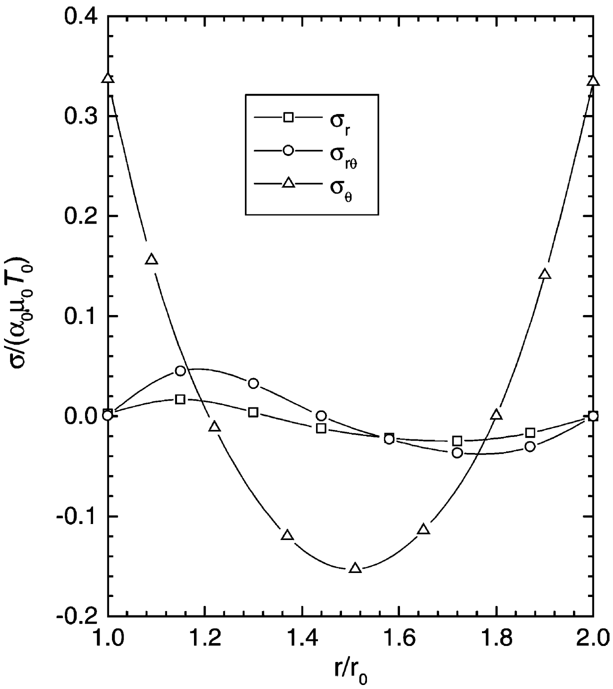


Fig. 2. Thermal stresses induced by $\theta$-dependent temperature ($n = 2$, $q = 0$).

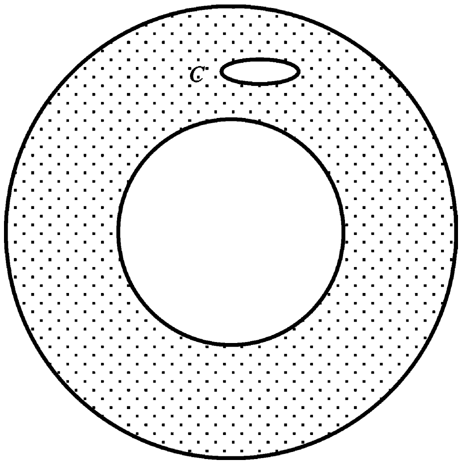


Fig. 3. A cross-section of a FGM hollow circular cylinder.

temperature field that induces no $\sigma_r$, $\sigma_{r\theta}$, or $\sigma_\theta$ is the trivial case, $T(r,\theta) = 0$. This conclusion can be generalized to other FGM configurations by the following simple reasoning.

In Fig. 3 a cross-section of a FGM hollow circular cylinder is shown. In the stress-free state, the cross-section can actually be looked upon as consisting of two cross-sections: one belonging to a cylindrical body with cross-section $C$, and the other to the hollow circular cylinder that is perforated by the cylindrical body bounded by $C$. As $T(r,\theta) = 0$ is the unique temperature distribution that causes no $\sigma_r$, $\sigma_{r\theta}$, or $\sigma_\theta$ in the FGM cylinder, this should be equally true for the two cylindrical bodies that are indicated above. Because contour $C$ can be of any shape and dimension, it follows that for any plane configuration of FGM, $T(r,\theta) = 0$ is the unique temperature distribution that does not cause stresses $\sigma_r$, $\sigma_{r\theta}$, or $\sigma_\theta$.

### 5.2. Numerical computation of thermal stresses

The numerical results that are determined from the theoretical solutions that were developed in the foregoing sections are discussed in this sub-section. Plane strain, $v = 1/3$, $p = 1/2$, $s = 1/2$, $r_0 = 1$, $r_N = 2$ and free traction boundary conditions on the inner and outer boundaries are assumed in all of the numerical results.

For uniform temperature $T_0$, thermal stresses $\sigma_r$ and $\sigma_\theta$ are shown in Fig. 1 for $q = 0$ and 0.25, and in Figs. 4 and 5 for $q = 0.5$ and 1 respectively. The result for $q = 0$ demonstrates that thermal stresses are set up in the FGM cylinder even when the material is homogeneous with respect to thermal expansion, i.e. when $\alpha(r) = \alpha_0 =$ constant. As expected, a larger $q$ brings about larger values of thermal stresses.

For temperatures with $\theta$-dependence, under the following temperature boundary conditions:

$$r = r_0, \quad T = 0, \qquad r = r_N, \quad T = T_0 \cos n\theta, \tag{86}$$

$\sigma_r$, $\sigma_{r\theta}$, and $\sigma_\theta$ are depicted in Figs. 6–8 ($n = 2$, $q = 0.25, 0.5, 1$), Figs. 9–11 ($n = 4$, $q = 0.25, 0.5, 1$) and Figs. 12–14 ($n = 8$, $q = 0.25, 0.5, 1$). Fig. 2 demonstrates the thermal stresses in the FGM cylinder for $n = 2$ and $q = 0$. In Figs. 2 and 6–14, the values of $\sigma_r$ and $\sigma_\theta$ are given for $\theta = 0$, and those of $\sigma_{r\theta}$ for $\theta = \pi/2n$.

From the numerical results one sees that $\sigma_\theta$ is the predominant stress, with its maxima in magnitude exceeding those of $\sigma_r$ and $\sigma_{r\theta}$ in all cases. Thermal expansion in the FGM cylinder is restrained by the non-homogeneous properties $\lambda(r)$ and $\alpha(r)$ in addition to temperature distributions. A small change in the thermal expansion index $q$ will bring about large changes in the magnitude of thermal stresses or the

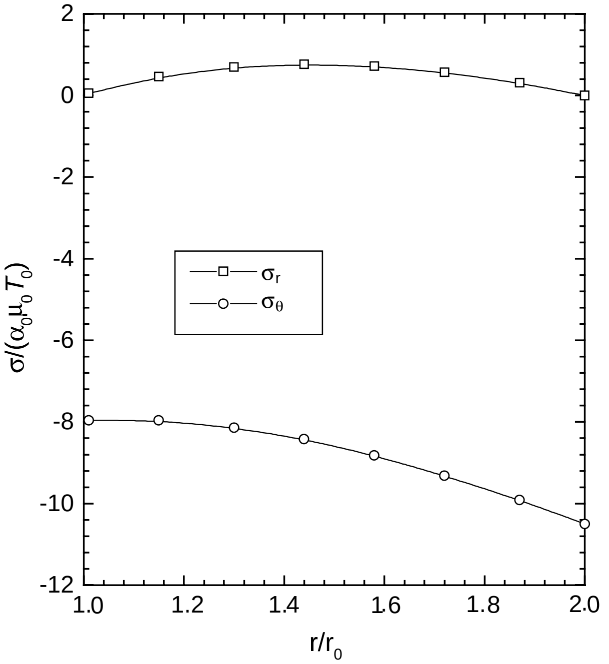


Fig. 4. Thermal stresses induced by uniform temperature ($q = 0.5$).

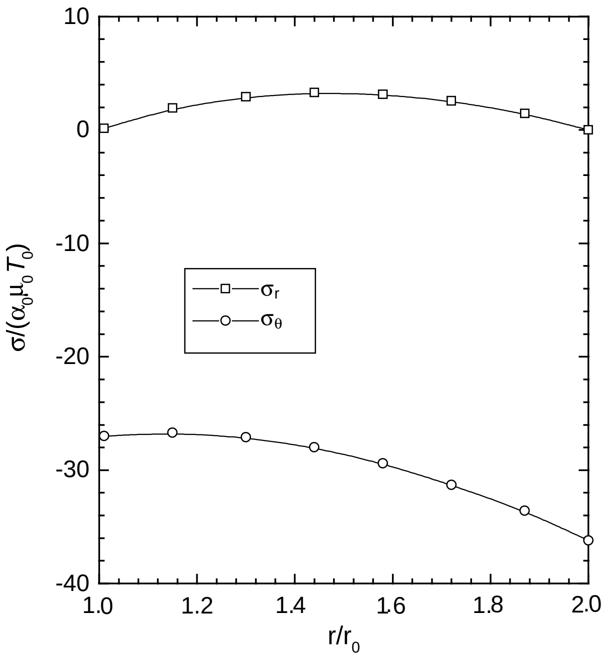


Fig. 5. Thermal stresses induced by uniform temperature ($q = 1$).

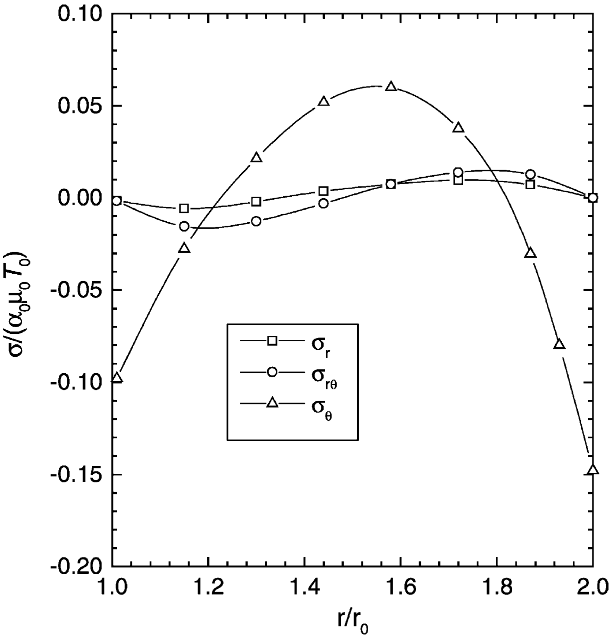


Fig. 6. Thermal stresses induced by $\theta$-dependent temperature ($n = 2$, $q = 0.25$).

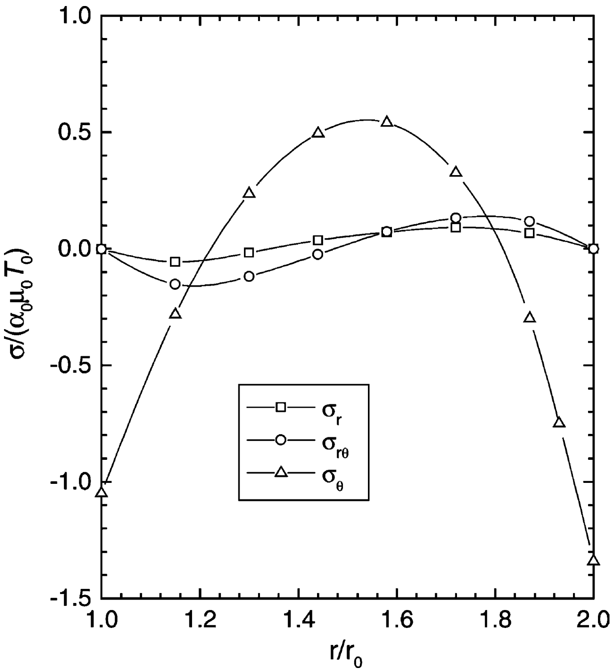


Fig. 7. Thermal stresses induced by $\theta$-dependent temperature ($n = 2$, $q = 0.5$).

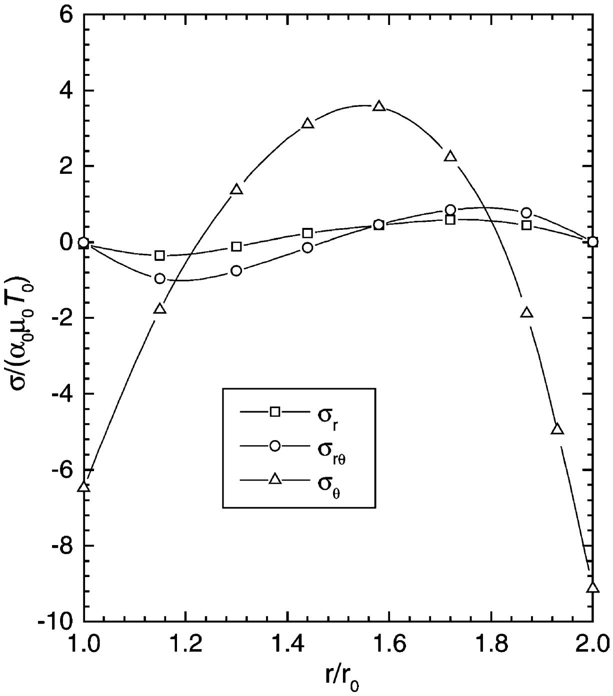


Fig. 8. Thermal stresses induced by $\theta$-dependent temperature ($n = 2$, $q = 1$).

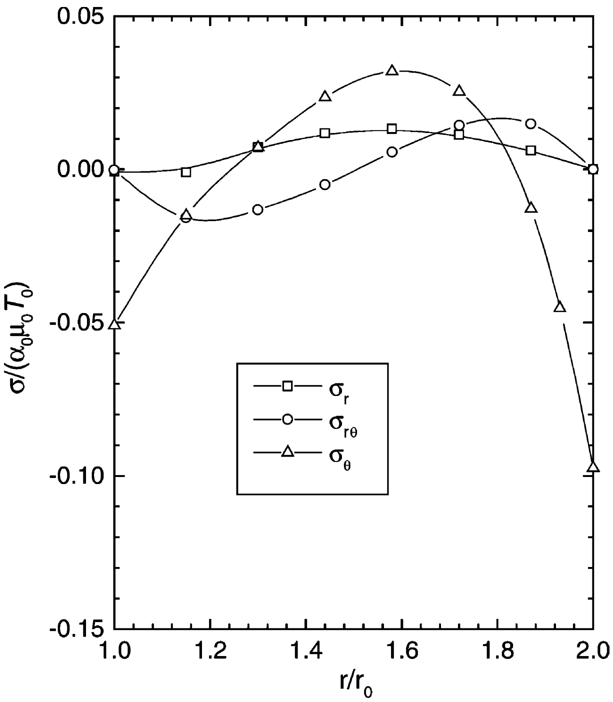


Fig. 9. Thermal stresses induced by $\theta$-dependent temperature ($n = 4$, $q = 0.25$).

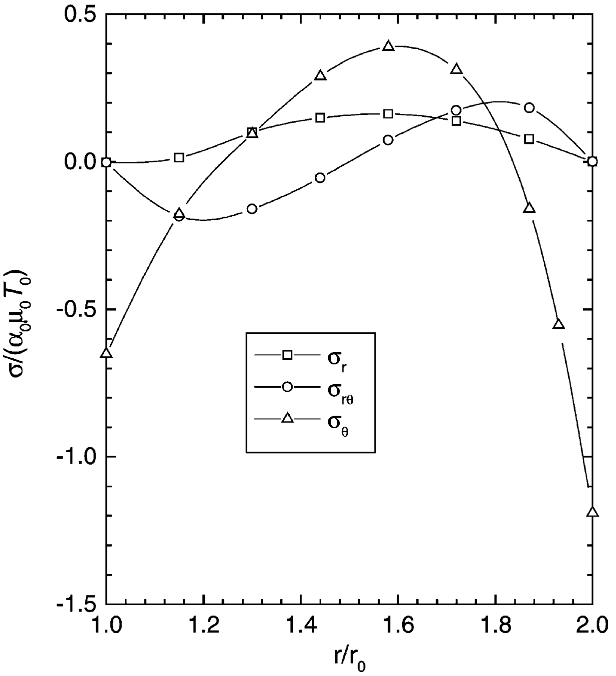


Fig. 10. Thermal stresses induced by $\theta$-dependent temperature ($n = 4$, $q = 0.5$).

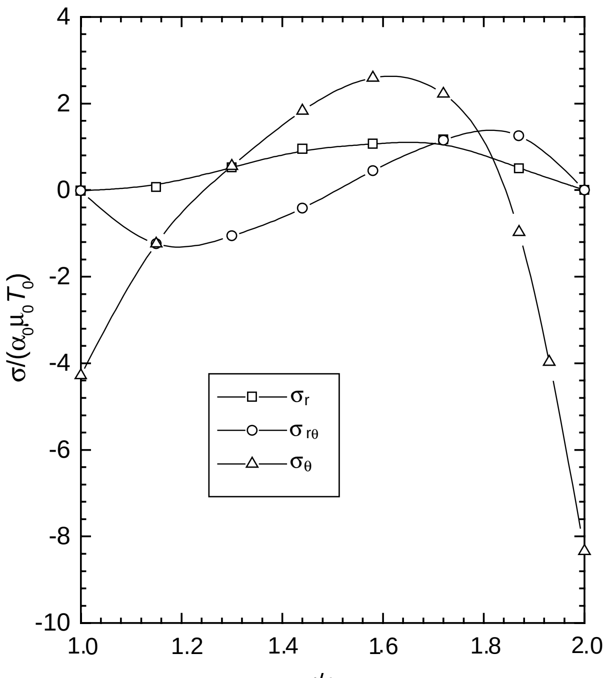


Fig. 11. Thermal stresses induced by $\theta$-dependent temperature ($n = 4$, $q = 1$).

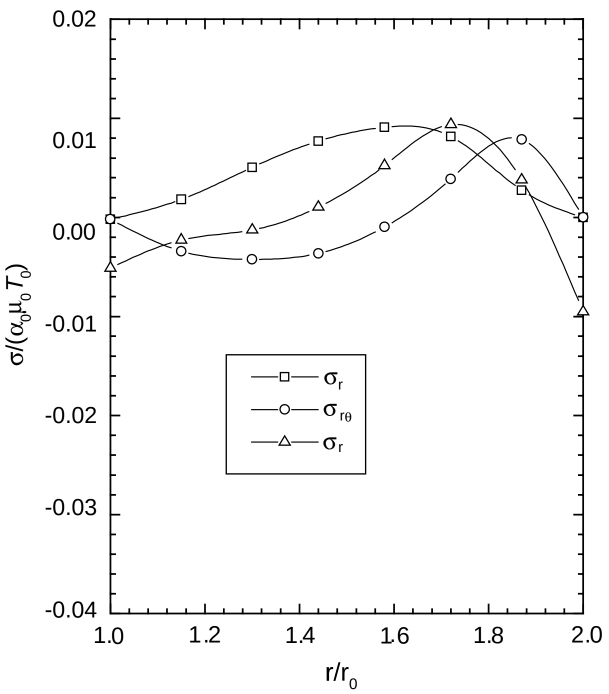


Fig. 12. Thermal stresses induced by $\theta$-dependent temperature ($n = 8$, $q = 0.25$).

reversal of direction, as indicated in Figs. 1, 2, 4 and 5. It follows that thermal stresses in the FGM cylinder are affected by more factors than are its homogeneous counterparts, and it is more complicated to evaluate.

The theoretical solutions are well suited to numerical manipulation, and provide accurate and reliable numerical results in the figures when 20–40 terms are included in the series.

## 5.3. General properties of thermal stresses

Some general properties can be drawn for thermal stresses in the FGM cylinder. The usefulness of a particular numerical result can be extended by using these general properties.

Under uniform temperature, the thermal stresses are dependent on, in addition to the variable $r$, the parameters $T_0$, $\alpha_0$, $\mu_0$, $q$, $s$, $r_0$, and $r_N$. For a fixed set of the parameters, the corresponding radial stress is denoted by $\sigma_r(r; T_0, \alpha_0, \mu_0, q, s, r_0, r_N)$. A simple examination of the solution shows that $\sigma_r$ is proportional to $T_0$. This fact can more precisely be expressed by the following formula:

$$\sigma_r(r; mT_0, \alpha_0, \mu_0, q, s, r_0, r_N) = m\sigma_r(r; T_0, \alpha_0, \mu_0, q, s, r_0, r_N), \quad (87)$$

where $m$ is a real number. Likewise, for $m$ to be positive, we have

$$\begin{aligned} &\sigma_r(r; T_0, m\alpha_0, \mu_0, q, s, r_0, r_N) = m\sigma_r(r; T_0, \alpha_0, \mu_0, q, s, r_0, r_N), \\ &\sigma_r(r; T_0, \alpha_0, m\mu_0, q, s, r_0, r_N) = m\sigma_r(r; T_0, \alpha_0, \mu_0, q, s, r_0, r_N), \\ &\sigma_r\left(\frac{r}{m}; T_0, \alpha_0, \mu_0, mq, ms, \frac{r_0}{m}, \frac{r_N}{m}\right) = \sigma_r(r; T_0, \alpha_0, \mu_0, q, s, r_0, r_N). \end{aligned} \quad (88)$$

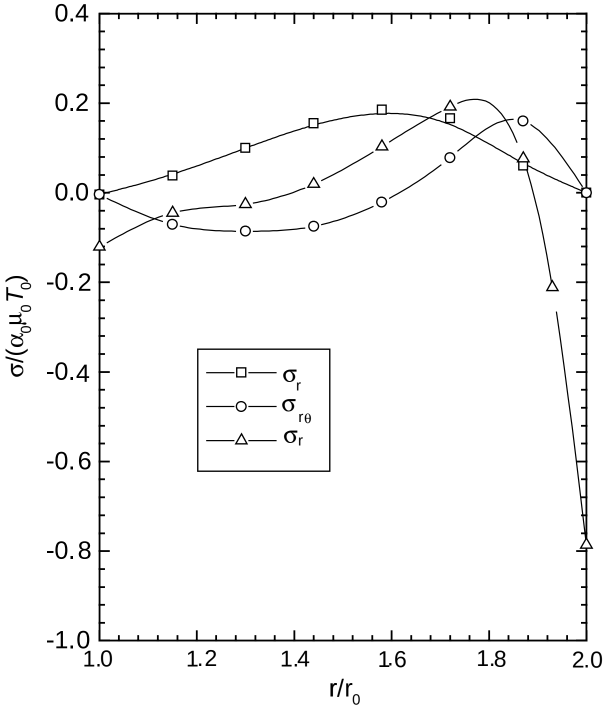


Fig. 13. Thermal stresses induced by $\theta$-dependent temperature ($n = 8$, $q = 0.5$).

The last formula in Eq. (88) indicates that $\sigma_r$ in a FGM cylinder at point $r$ equals that in another FGM cylinder, which is $1/m$ times larger or smaller in geometry and $m$ times in $q$ and $s$, at the geometrically similar point $r/m$.

When the temperature is $\theta$-dependent, $\sigma_r$ depends on the parameters $T_0$, $\alpha_0$, $\mu_0$, $q$, $s$, $r_0$, $r_N$, $n$, and $p$. We then have the following general properties:

$$\sigma_r(r; mT_0, \alpha_0, \mu_0, q, s, r_0, r_N, n, p) = m\sigma_r(r; T_0, \alpha_0, \mu_0, q, s, r_0, r_N, n, p), \tag{89}$$

where $m$ is real, and for $m$ to be positive

$$\begin{aligned} &\sigma_r(r; T_0, m\alpha_0, \mu_0, q, s, r_0, r_N, n, p) = m\sigma_r(r; T_0, \alpha_0, \mu_0, q, s, r_0, r_N, n, p), \\ &\sigma_r(r; T_0, \alpha_0, m\mu_0, q, s, r_0, r_N, n, p) = m\sigma_r(r; T_0, \alpha_0, \mu_0, q, s, r_0, r_N, n, p), \\ &\sigma_r\left(\frac{r}{m}; T_0, \alpha_0, \mu_0, mq, ms, \frac{r_0}{m}, \frac{r_N}{m}, n, mp\right) = \sigma_r(r; T_0, \alpha_0, \mu_0, q, s, r_0, r_N, n, p). \end{aligned} \tag{90}$$

Eqs. (87)–(90) remain true when $\sigma_r$ is replaced by $\sigma_{r\theta}$ or $\sigma_\theta$. These analytic observations have been confirmed by numerical verification.

## 5.4. Extension of the solutions

The thermal stress solutions are obtained for the steady-state temperature distributions of series forms (12) and (18). By replacing the coefficients in the series with suitable constants, the resultant series can represent the transient temperature distributions at a certain time $t$. The developed solutions can then be used to evaluate the transient thermal stresses in FGM cylinders.

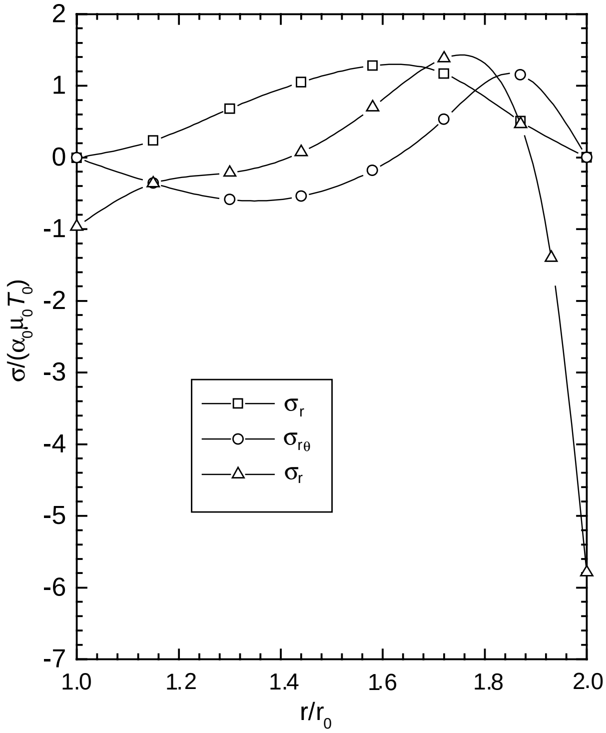


Fig. 14. Thermal stresses induced by $\theta$-dependent temperature ($n = 8$, $q = 1$).

The development is based on a plane strain condition. The result for generalized plane stress can be obtained by letting $\kappa = (3 - v)/(1 + v)$ and introducing a few other amendments that are well known in the theory of elasticity.

## 6. Conclusions

This article presents an analysis of the temperature and thermal stresses in a FGM hollow circular cylinder. It also provides a new solution method for thermal stresses in homogeneous cylinders as a special case. By using the solutions, particular digital values can be obtained, and systematic parameter study can be carried out with simple numerical manipulations. The case of $\theta$-dependence, which appears to be untouched in the existing literature, is covered in the solutions. The solutions are obtained by a novel approach: the matching of the homogeneous solutions with ingenuous propositions for the solution form in the semi-inverse method. These results will be useful for future reference.


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